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# A STUDY OF THE STABILITY OF REINFORCED CYLINDRICAL AND CONICAL SHELLS SUBJECTED TO VARIOUS TYPES AND COMBINATIONS OF LOADS

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*Huntsville, Alabama*

## SECTION III - General Instability of an Orthotropic Circular Conical Shell Subjected To Hydrostatic Pressure and A Compressive Axial Force

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# GENERAL INSTABILITY OF AN ORTHOTROPIC CIRCULAR CONICAL SHELL SUBJECTED TO HYDROSTATIC PRESSURE AND A COMPRESSIVE AXIAL FORCE

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## INTRODUCTION

The need for weight saving in a cylindrical or conical missile body section has resulted in the use of ring stiffeners and axial stiffeners to insure structural integrity and stability. This need has also resulted in the development of single-faced or double faced corrugated construction and in the use of ribbed stiffened monolithic structures fabricated by milling.

A circular conical shell manufactured as mentioned above can be considered by representing it by an equivalent orthotropic shell. The general Donnell type instability differential equation for such a shell can be determined by an energy analysis similar to the procedure used in Reference (1) for an orthotropic cylindrical shell.

In the present paper a set of instability equilibrium equations for an orthotropic circular conical shell are derived by applying variational methods to the expression for the total energy of the shell. From these equilibrium equations an eighth order differential equation of the Donnell type is obtained for a circular conical shell segment of uniform thickness subjected to an external hydrostatic pressure and a compressive axial force.

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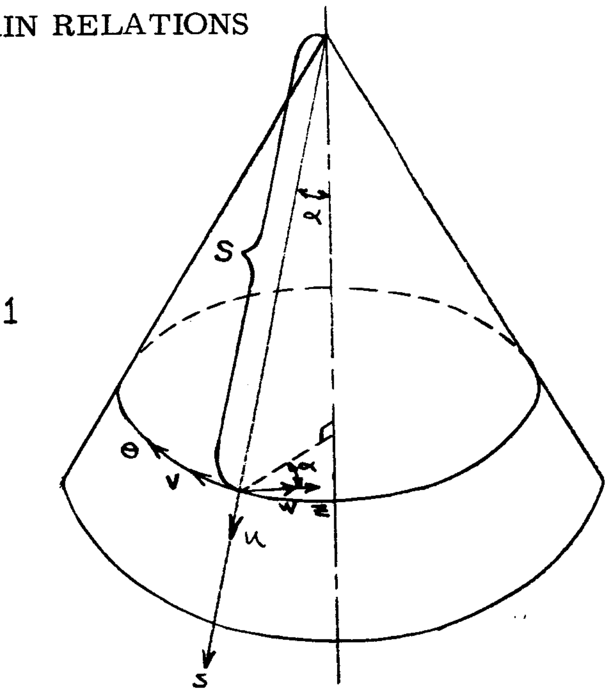
STRESS-STRAIN RELATIONS

Fig 1




The circular conical shell geometry is shown in Figure (1) together with the coordinate system used and the corresponding middle-surface displacements. In terms of the shell middle-surface displacements, $u$, $v$, and $w$, the expressions used for the buckling strains in the shell wall are written as follows:

$$
\begin{aligned}
e_{ss} &= u_{,s} + \left(\tfrac{1}{2}\right)\left[ \tfrac{1}{s^2}\left(v^2 + w^2\right) + u_{,s}^2 + v_{,s}^2 + w_{,s}^2 \right] - \tfrac{1}{s}\left[ v\, v_{,s} + w\, w_{,s}\right) - z\, w_{,ss} \\
e_{\theta\theta} &= \tfrac{1}{s}\left[ u - v_{,\theta} \csc\alpha - w \cot\alpha \right] + \left(\csc\alpha / s^2\right)\left(u\, v_{,\theta} - u\, w \cos\alpha - v_{,\theta}\, w \cot\alpha\right) \\
&\quad + \left(\csc^2 \alpha / 2s^2\right)\left(u^2 \sin^2\alpha + w^2 \cos^2\alpha + u_{,\theta}^2 + v_{,\theta}^2 + w_{,\theta}^2\right) - \tfrac{z}{s^2}\left(s\, w_{,s} + w_{,\theta\theta} \csc^2\alpha\right) \\
e_{s\theta} &= \tfrac{1}{s}\left(u_{,\theta} \csc\alpha - v + s\, v_{,s}\right) + \left(\csc \tfrac{\alpha}{s}\right)\left(u_{,s}\, u_{,\theta} + v_{,s}\, v_{,\theta} + w_{,s}\, w_{,\theta} - 2 v_{,s}\, w \cos\alpha\right) \\
&\quad - 2z\left(\csc \tfrac{\alpha}{s^2}\right)\left( s\, w_{,s\theta} - w_{,\theta}\right)
\end{aligned}
\qquad (1)
$$

where $e_{ss}$, $e_{\theta\theta}$, and $e_{s\theta}$, are the generatrix, circumferential, and shear strains, respectively; and a comma indicates differentiation with respect to the succeeding variables.

For a homogenous orthotropic material, the stress-strain relations in generalized plane stress can be written as

$$\sigma_{ss} = \frac{E_s(e_{ss} + \nu_{\theta s} e_{\theta\theta})}{1 - \nu_{\theta s}\nu_{s\theta}} \qquad \sigma_{\theta\theta} = \frac{E_\theta(e_{\theta\theta} + \nu_{\theta s} e_{ss})}{1 - \nu_{s\theta}\nu_{\theta s}}$$

$$\sigma_{s\theta} = G e_{s\theta} \tag{2}$$

In the above equations $E_s$ and $E_\theta$ are the values of the moduli of elasticity averaged over the thickness in the generatrix and circumferential directions, respectively; $G$ is the average shear modulus, and $\nu_{s\theta}$ and $\nu_{\theta s}$ are Poisson's ratios.

For convenience in later calculations, the following constants and notations, similar to those given in Reference (1), are introduced:

$$\alpha_1 = \frac{E_s h}{1 - \nu_{s\theta}\nu_{\theta s}} \qquad \alpha_2 = \frac{E_\theta h}{1 - \nu_{s\theta}\nu_{\theta s}} \qquad \alpha_3 = Gh$$

$$D_1 = \frac{E_s h^3}{12(1 - \nu_{s\theta}\nu_{\theta s})} \qquad D_2 \frac{E_\theta h}{12(1 - \nu_{s\theta}\nu_{\theta s})} \qquad D_3 = \frac{Gh^3}{12} \tag{3}$$

where $h$ is the thickness of the shell. The following stress resultants are also defined:

$$\bar{N}_{ss} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{\sigma}_{ss}\, dz \qquad \bar{N}_{\theta\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{\sigma}_{\theta\theta}\, dz \qquad \bar{N}_{s\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{\sigma}_{s\theta}\, dz \tag{4}$$

Based on Maxwell's reciprocal theorem, the following relationship must hold between the elastic constants:

$$\frac{\nu_{s\theta}}{E_s} = \frac{\nu_{\theta s}}{E_\theta}. \tag{5}$$

## STRAIN ENERGY AND TOTAL ENERGY EXPRESSIONS

The instability differential equations of equilibrium will be derived using the same procedure as given in Reference (1). For an elastic system, a criterion of buckling is that the variation of the change in the energy of the system due to buckling, with respect to the displacements, must be zero. Described mathematically, this criterion becomes

$$\delta (U + V) = 0 \tag{6}$$

where $U$ is the change in the strain energy of the shell and $V$ is the change in the potential energy of the external forces during the buckling process.

If initial bending stresses are neglected

$$U = \int_{V_s} \left[ \left( \bar{\sigma}_{ss} e_{ss} + \bar{\sigma}_{\theta\theta} e_{\theta\theta} + \bar{\sigma}_{s\theta} e_{s\theta} \right) + \frac{1}{2} \left( \sigma_{ss} e_{ss} + \sigma_{\theta\theta} e_{\theta\theta} + \sigma_{s\theta} e_{s\theta} \right) \right] dV_s \tag{7}$$

In the preceding expression $\bar{\sigma}_{ss}$, $\bar{\sigma}_{\theta\theta}$, and $\bar{\sigma}_{s\theta}$ are the membrane stresses existing in the shell in the compressed but unbuckled state; $\sigma_{ss}$, $\sigma_{\theta\theta}$, and $\sigma_{s\theta}$ are the stresses superimposed during the buckling process, and $V_s$ is the volume of the shell wall.

For a circular cone frustum with an applied hydrostatic pressure and an axial compressive force $Q$, the following membrane stresses, obtained from References (3) and (4), are used in this analysis:

$$\bar{N}_{ss} = - \frac{ps \tan \alpha}{2} - \frac{Q}{\pi s \sin 2\alpha}$$

$$\bar{N}_{\theta\theta} = - ps \tan \alpha$$

$$N_{s\theta} = 0 \tag{8}$$

The preceding expressions are based on the assumptions that the boundary conditions, such as ring-stiffened edges, do not permit pre-buckling bending or shear stresses to exist.

Based on the preceding limitations and assumptions the change in the potential energy of the external forces during the buckling process is:

$$V = -\int_{V_s} p\,dV_c - \int_0^{2\pi} \left[ \bar{N}_{ss}\, s\, u \right]_{s_1}^{s_0} \sin\alpha \; d\theta \tag{9}$$

where $V_c$ is the volume enclosed by the shell and $s_1$ and $s_0$ are the boundary values of $s$ for circular cone frustum. The first integral in the preceding expression, which is the pressure times the shell volume change, is the work done by the hydrostatic pressure $p$. The second integral is the work done by the stress resultants in the generatrix direction as a result of the boundary displacements at the edges of the cone frustum during the buckling process.

From the geometry of the shell the following relations are obtained:

$$dV_s = s \sin\alpha \; d\theta\, ds\, dz \qquad\qquad dA_s = s \sin\alpha \; d\theta\, ds \tag{10}$$

where $A_s$ is the area of the middle surface of the shell.

By combining Equations (2), (4), (5), (7), (9), and (10), the expression for the total energy $U + V$ can be written as follows:

$$\begin{aligned} U + V = &\int_{s_1}^{s_0}\int_0^{2\pi} \left[ \left( \bar{N}_{ss}\, e_{ss} + \bar{N}_{\theta\theta}\, e_{\theta\theta} + \bar{N}_{s\theta}\, e_{s\theta} \right)\left( s \sin\alpha \right) \right] d\theta\, ds \\ &+ \frac{1}{2}\int_{-\frac{h}{2}}^{h/2}\int_{s_1}^{s_0}\int_0^{2\pi} \left[ \left( 1 - \nu_{\theta s}\nu_{s\theta} \right)^{-1} \left( E_s e_{ss}^2 + E_\theta e_{\theta\theta}^2 \right.\right. \\ &\left.\left. + 2\nu_{\theta s} E_s\, e_{ss}\, e_{\theta\theta} + G\, e_{s\theta}^2 \right] \left[ s \sin\alpha \right] d\theta\, ds\, dz \\ &- \int_{V_s} p\, dV_c - \int_0^{2\pi} \left[ \bar{N}_{ss}\, s u \right]_{s_1}^{s_0} \left[ \sin\alpha \right] d\theta \end{aligned} \tag{11}$$

Also from the geometry of the shell the following expression is obtained for the work done by the hydrostatic pressure:

$$
\begin{aligned}
\int_{V_s} p\,dV_c = p\int_{s_1}^{s_0}\int_0^{2\pi}\Big\{\tfrac{1}{4}\big[&4su\sin^2\alpha\sec\alpha + 2s^2u_{,s}\sin^2\alpha\sec\alpha \\
&+2sv_{,\theta}\tan\alpha - 4sw\sin\alpha + 3u^2\sin^2\alpha\sec\alpha + 4suu_{,s}\sin^2\alpha\sec\alpha \\
&+2su_{,s}v_{,\theta}\tan\alpha + 4uv_{,\theta}\tan\alpha - 6uw\sin\alpha - 4su_{,s}w\sin\alpha \\
&-4v_{,\theta}w + v^2\sin^2\alpha\sec\alpha - 2svv_{,s}\sin^2\alpha\sec\alpha + 3w^2\cos\alpha \\
&+w^2\sin^2\alpha\sec\alpha - 2sww_{,s}\sin^2\alpha\sec\alpha + \sec\alpha(u_{,\theta}^2 + v_{,\theta}^2 + w_{,\theta}^2) \\
&+(s^2\sin^2\alpha\sec\alpha)(u_{,s}^2 + v_{,s}^2 + w_{,s}^2)\big]\Big\}\,d\theta\,ds \qquad (12)
\end{aligned}
$$

In the preceding expression and in the succeeding development the terms involving the displacement $v$ are considered negligible and are omitted.

By substituting Equations (1), (3), (5), (8), and (12) into Equation (11), and retaining only terms up to the second order, the following expression for the total energy $\mathcal{U}+\mathcal{V}$ is obtained in terms of the displacements $u$, $v$, $w$, and their derivatives:

$$
\begin{aligned}
\mathcal{U}+\mathcal{V} = \tfrac{1}{2}\int_{s_1}^{s_0}\int_0^{2\pi}\Big\{&\alpha_1\big[su_{,s}^2\sin\alpha + 2\nu_{\theta s}(uu_{,s}\sin\alpha + u_{,s}v_{,\theta} - u_{,s}w\cos\alpha)\big] \\
&+\alpha_2/s\,\big[u^2\sin\alpha + 2uv_{,\theta} - 2uw\cos\alpha + v_{,\theta}^2\sec\alpha - 2v_{,\theta}w\cot\alpha \\
&+w^2\cos\alpha\cot\alpha\big] + \alpha_3/s\,\big[u_{,\theta}^2\csc\alpha - 2u_{,\theta}v + 2su_{,\theta}v_{,s} + v^2\sin\alpha \\
&-2sv_{,s}v\sin\alpha + s^2v_{,s}^2\sin\alpha\big] + D_1/s\,\big[s^2w_{,ss}^2\sin\alpha + 2\nu_{\theta s}w_{,ss}w_{,\theta\theta}\csc\alpha \\
&+2\nu_{\theta s}sw_{,s}w_{,ss}\sin\alpha\big] + D_2/s^3\,\big[s^2w_{,s}^2\sin\alpha + 2sw_{,s}w_{,\theta\theta}\csc\alpha \\
&+w_{,\theta\theta}^2\csc^3\alpha\big] + 4D_3/s^3\big[s^2w_{,s\theta}^2\csc\alpha - 2sw_{,s\theta}w_{,\theta}\csc\alpha + w_{,\theta}^2\csc\alpha\big] \\
&+p/4\,\big[8sw\sin\alpha - 8su\sin^2\alpha\sec\alpha - 6sv_{,\theta}\tan\alpha - 4s^2u_{,s}\sin^2\alpha\sec\alpha \\
&-5u^2\sin^2\alpha\sec\alpha - 4suu_{,s}\sin^2\alpha\sec\alpha - 2su_{,s}v_{,\theta}\tan\alpha \\
&-8uv_{,\theta}\tan\alpha + 10uw\sin\alpha + 4su_{,s}w\sin\alpha - 2v^2\sin^2\alpha\sec\alpha \\
&+4svv_{,s}\sin^2\alpha\sec\alpha + 8v_{,\theta}w - 3w^2\cos\alpha - 4w^2\sin^2\alpha\sec\alpha \\
&+4sww_{,s}\sin^2\alpha\sec\alpha - 2(s^2\sin^2\alpha\sec\alpha)(u_{,s}^2 + v_{,s}^2 + w_{,s}^2) \\
&-3\sec\alpha(u_{,\theta}^2 + v_{,\theta}^2 + w_{,\theta}^2)\big] - (Q\sec\alpha/\pi s^2)\big[\tfrac{1}{2}(v^2 + w^2 + s^2u_{,s}^2 \\
&+s^2v_{,s}^2 + s^2w_{,s}^2) + s^2u_{,s} - svv_{,s} \cancel{- sww_{,s}}\big]\Big\}\,d\theta\,ds \\
&+\tfrac{1}{2}\int_0^{2\pi}\Big[ps^2u\sin^2\alpha\sec\alpha + \tfrac{1}{\pi}Qu\sec\alpha\Big]_{s_1}^{s_0}\,d\theta \qquad (13)
\end{aligned}
$$

EQUILIBRIUM EQUATIONS AND NATURAL BOUNDARY CONDITIONS RESULTING FROM THE APPLICATION OF VARIATIONAL PROCEDURES

The following expression for the variation in the total energy is obtained from Equation (13):

$$
\begin{aligned}
\delta(U+V) = \int_{s_1}^{s_0}\int_0^{2\pi} \Big\{ & \Big[\alpha_1(\nu_{\theta s} u_{,s}\sin\alpha) + \tfrac{\alpha_2}{s}(u\sin\alpha + v_{,\theta} - w\cos\alpha) + \tfrac{p}{2}(s w\sin\alpha \\
& - 4v_{,\theta}\tan\alpha - s u\sin^2\alpha\sec\alpha - 2s u_{,s}\sin^2\alpha\sec\alpha)\Big]\delta u \\
& + \Big[\alpha_1(s u_{,s}\sin\alpha + \nu_{\theta s} u\sin\alpha + \nu_{\theta s} v_{,\theta} - \nu_{\theta s} w\cos\alpha) - \tfrac{1}{2\pi}(Q u_{,s}\sec\alpha) \\
& + \tfrac{p}{2}(\sec\alpha)(2s w\sin\alpha\cos\alpha - 2s u\sin^2\alpha - s v_{,\theta}\sin\alpha - 2s^2 u_{,s}\sin^2\alpha)\Big]\delta u_{,s} \\
& + \Big[\tfrac{\alpha_3}{s}(u_{,\theta}\csc\alpha - v + s v_{,s}) - \tfrac{p}{2}(3u_{,\theta}\sec\alpha)\Big]\delta u_{,\theta} + \Big[\tfrac{\alpha_3}{s}(v\sin\alpha - u_{,\theta} \\
& - s v_{,s}\sin\alpha) + (p\sec\alpha\sin^2\alpha)(s v_{,s} - v) - (Q\sec\alpha/2\pi s^2)(v - s v_{,s})\Big]\delta v \\
& + \Big[\alpha_3(u_{,\theta} - v\sin\alpha + s v_{,s}\sin\alpha) + (p\sec\alpha\sin^2\alpha)(s v - s^2 v_{,s}) \\
& - (Q\sec\alpha/2\pi s)(s v_{,s} - v)\Big]\delta v_{,s} + \Big[\alpha_1(\nu_{\theta s} u_{,s}) + \tfrac{\alpha_2}{s}(u + v_{,\theta}\csc\alpha \\
& - w\cot\alpha) + \tfrac{p}{2}(4w - s u_{,s}\tan\alpha - 4u\tan\alpha - 3v_{,\theta}\sec\alpha)\Big]\delta v_{,\theta} \\
& + \Big[\alpha_1(-\nu_{\theta s} u_{,s}\cos\alpha) + \tfrac{\alpha_2}{s}(w\cos^2\alpha\csc\alpha - u\cos\alpha - v_{,\theta}\cot\alpha) \\
& + \tfrac{p}{2}(s u\sin\alpha + 2s u_{,s}\sin\alpha + 4v_{,\theta} + 3w\cos\alpha + 4w\sin^2\alpha\sec\alpha \\
& + 2s w_{,s}\sin^2\alpha\sec\alpha) - (Q\sec\alpha/2\pi s^2)(w - s w_{,s})\Big]\delta w \\
& + \Big[D_1(\nu_{\theta s} w_{,ss}\sin\alpha) + \tfrac{D_2}{s^2}(s w_{,s}\sin\alpha + w_{,\theta\theta}\csc\alpha) + (p\sec\alpha\sin^2\alpha) \\
& (s w - s^2 w_{,s}) - (Q\sec\alpha/2\pi s)(s w_{,s} - w)\Big]\delta w_{,s} \\
& + \Big[(2D_3\csc\alpha/s^3)(2w_{,\theta} - 2s w_{,s\theta}) - \tfrac{1}{2}(p\sec\alpha)(3w_{,\theta})\Big]\delta w_{,\theta} \\
& + \Big[\tfrac{D_1}{s}(s^2 w_{,ss}\sin\alpha + \nu_{\theta s} s w_{,s}\sin\alpha + \nu_{\theta s} w_{,\theta\theta}\csc\alpha)\Big]\delta w_{,ss} \\
& \Big[(4D_3\csc\alpha/s^2)(s w_{,s\theta} - w_{,\theta})\Big]\delta w_{,s\theta} + \Big[(D_1\csc\alpha/s)(\nu_{\theta s} w_{,ss}) \\
& + (D_2\csc\alpha/s^3)(s w_{,s} + w_{,\theta\theta}\csc^2\alpha)\Big]\delta w_{,\theta\theta}\Big\}\, d\theta\, ds \\
& + \tfrac{1}{2}\int_0^{2\pi}\Big[p s^2\sin^2\alpha\sec\alpha + \tfrac{1}{\pi}(Q\sec\alpha)\Big]_{s_1}^{s_0}\delta u\, d\theta
\end{aligned}
\tag{14}
$$

Integration of Equation (14) by parts results in the following expression:

$$
\begin{aligned}
\delta(U+V) = {} & \int_{s_1}^{s_0}\int_0^{2\pi}\Big\{\Big[\alpha_1(-u_{,s}\sin\alpha - s u_{,ss}\sin\alpha - \nu_{\theta s} v_{,\theta s} + \nu_{\theta s} w_{,s}\cos\alpha) \\
& + \alpha_2/s(u\sin\alpha + v_{,\theta} - w\cos\alpha) - \alpha_3/s(u_{,\theta\theta}\csc\alpha - v_{,\theta} + s v_{,s\theta}) + P/2(-3u\sin^2\alpha\sec\alpha \\
& + 4s u_{,s}\sin^2\alpha\sec\alpha + 2s^2 u_{,ss}\sin^2\alpha\sec\alpha + 3u_{,\theta\theta}\sec\alpha - 3v_{,\theta}\tan\alpha \\
& + s v_{,\theta s}\tan\alpha + 3w\sin\alpha - 2s w_{,s}\sin\alpha) + (Q\sec\alpha/2\pi)(u_{,ss})\Big]\delta u \\
& + \Big[-\alpha_1(\nu_{\theta s} u_{,s\theta}) - \alpha_3/s(u_{,\theta} + v_{,\theta\theta}\csc\alpha - w_{,\theta}\cot\alpha) - \alpha_3/s(u_{,\theta} + s u_{,\theta s} \\
& - v\sin\alpha + s v_{,s}\sin\alpha + s^2 v_{,ss}\sin\alpha) + P/2(4u_{,\theta}\tan\alpha + s u_{,s\theta}\tan\alpha \\
& - 4\sin^2\alpha\sec\alpha + 4s v_{,s}\sin^2\alpha\sec\alpha + 2s^2 v_{,ss}\sin^2\alpha\sec\alpha + 3v_{,\theta\theta}\sec\alpha - 4w_{,\theta}) \\
& + (Q\sec\alpha/2\pi)(v_{,ss})\Big]\delta v + \Big[-\alpha_1(\nu_{\theta s} u_{,s}\cos\alpha) - \alpha_2/s(u\cos\alpha + v_{,\theta}\cot\alpha \\
& - w\cos^2\alpha\sec\alpha) + P/2(5u\sin\alpha + 2s u_{,s}\sin\alpha + 4v_{,\theta} + 3w\cos\alpha + 2w\sin^2\alpha\sec\alpha \\
& + 4s w_{,s}\sin^2\alpha\sec\alpha + 2s^2 w_{,ss}\sin^2\alpha\sec\alpha + 3w_{,\theta\theta}\sec\alpha) \\
& + D_1/s^3(2\nu_{\theta s} w_{,\theta\theta}\csc\alpha + 2s^3 w_{,sss}\sin\alpha - 2\nu_{\theta s} s w_{,\theta\theta s}\csc\alpha \\
& + s^4 w_{,ssss}\sin\alpha + 2\nu_{\theta s} s^2 w_{,\theta\theta ss}\csc\alpha) + (D_2/s^3)(s w_{,s}\sin\alpha - s^2 w_{,ss}\sin\alpha \\
& + 2w_{,\theta\theta}\csc\alpha + w_{,\theta\theta\theta\theta}\csc\alpha) + (4D_3/s^3)(w_{,\theta\theta} - s w_{,s\theta\theta} + s^2 w_{,ss\theta\theta})\csc\alpha \\
& + (Q\sec\alpha/2\pi) w_{,ss}\Big]\delta w\Big\}\, d\theta\, ds \\
& + \int_0^{2\pi}\Big\{\Big[\alpha_1(s u_{,s}\sin\alpha + \nu_{\theta s} u\sin\alpha + \nu_{\theta s} v_{,\theta} - \nu_{\theta s} w\cos\alpha) \\
& + (P/2)(-2s u\sin^2\alpha\sec\alpha - 2s^2 u_{,s}\sin^2\alpha\sec\alpha - s v_{,\theta}\tan\alpha \\
& + 2s w\sin\alpha + s^2\sin^2\alpha\sec\alpha) - (Q\sec\alpha/2\pi)(u_{,s} - 1)\Big]\delta u \\
& + \Big[\alpha_3(u_{,\theta} - v\sin\alpha + s v_{,s}\sin\alpha) + (ps\sin^2\alpha\sec\alpha)(v - s v_{,s}) \\
& - (Q\sec\alpha/2\pi s)(s v_{,s} - v)\Big]\delta v + \Big[(D_1/s^2)(-s^2 w_{,ss}\sin\alpha \\
& + \nu_{\theta s} w_{,\theta\theta}\csc\alpha - s^3 w_{,sss}\sin\alpha - s\nu_{\theta s} w_{,\theta\theta s}\sec\alpha) \\
& + (D_2/s^2)(s w_{,s}\sin\alpha + w_{,\theta\theta}\sec\alpha) + (4D_3/s^2)(\csc\alpha)(w_{,\theta\theta} - s w_{,s\theta\theta}) \\
& + (ps\sin^2\alpha\sec\alpha)(w - s w_{,s}) + (Q\sec\alpha/2\pi s)(w - s w_{,s})\Big]\delta w \\
& + \Big[(D_1/s)(s^2 w_{,ss}\sin\alpha + \nu_{\theta s} s w_{,s}\sin\alpha + \nu_{\theta s} w_{,\theta\theta}\csc\alpha)\Big]\delta w_{,s}\Big\}_{s_1}^{s_0} d\theta \\
& + \int_{s_1}^{s_0}\Big\{\Big[(\alpha_3/s)(u_{,\theta}\csc\alpha - v + s v_{,s}) - (P/2)(3u_{,\theta}\sec\alpha)\Big]\delta u \\
& + \Big[\alpha_1(\nu_{\theta s} u_{,s}) + (\alpha_2/s)(u + v_{,\theta}\csc\alpha - w\cot\alpha) - (P/2)(4u\tan\alpha \\
& + s u_{,s}\tan\alpha + 3v_{,\theta}\sec\alpha - 4w)\Big]\delta v + \Big[(D_1/s)(-\nu_{\theta s} w_{,ss\theta}\csc\alpha) \\
& + (D_2/s^3)(s w_{,s\theta}\csc\alpha + w_{,\theta\theta\theta}\csc^3\alpha) - (4D_3/s^3)(\csc\alpha)(w_{,\theta} - s w_{,s\theta} + s^2 w_{,ss\theta})
\end{aligned}
\qquad (15)
$$

$$-(P/2)(3W_{,\theta}\sec\alpha)]\delta W+[(D_1/s)(-\nu_{\theta s}W_{,ss}\csc\alpha)+(D_2/s^3)(sW_{,s}\csc\alpha+W_{,\theta\theta}\csc^3\alpha)]\,\delta W_{,\theta}\}_0^{2\pi}\,ds+(4D_3/s^2)(\csc\alpha)(sW_{,s\theta}-W_{,\theta})]_{s_1}^{s_0}\,]_0^{2\pi}\,\delta W \tag{15}$$

The changes in the total energy of the cone must vanish for any arbitrary virtual displacements when the system is in equilibrium. As a result the following three equations of equilibrium are obtained from Equation (15):

$$\begin{aligned}
&(\alpha_1/s)[(su_{,s}+s^2u_{,ss})(\sin\alpha)+\nu_{\theta s}(sV_{,\theta s}-sW_{,s}\cos\alpha)]\\
&-(\alpha_2/s)[(u\sin\alpha+V_{,\theta}-W\cos\alpha)+(\alpha_3/s)[u_{,\theta\theta}\csc\alpha-V_{,\theta}+sV_{,s\theta}]\\
&+(P/2)[(3u-4su_{,s}-2s^2u_{,ss})(\sin^2\alpha\sec\alpha)-3u_{,\theta\theta}\sec\alpha\\
&+(3V_{,\theta}-sV_{,\theta s})(\tan\alpha)-(3W-2sW_{,s})(\sin\alpha)]-(Q\sec\alpha/2\pi)[u_{,ss}]=0
\end{aligned} \tag{16}$$

$$\begin{aligned}
&(\alpha_1/s)[-\nu_{\theta s}su_{,s\theta}]-(\alpha_2/s)[(u_{,\theta}+V_{,\theta\theta}\csc\alpha-W_{,\theta}\cot\alpha)]\\
&-(\alpha_3/s)[u_{,\theta}+su_{,s\theta}+(-V+sV_{,s}+s^2V_{,ss})\sin\alpha]\\
&+(P/2)[(4u_{,\theta}+su_{,s\theta})\tan\alpha+(\sin^2\alpha\sec\alpha)(-4V+4sV_{,s}+2s^2V_{,ss}\\
&+3V_{,\theta\theta}\csc^2\alpha-4W_{,\theta})]+(Q\sec\alpha/2\pi)[V_{,ss}]\\
&+(Q\sec\alpha/2\pi s^2)[2V-2sV_{,s}+s^2V_{,ss}]=0
\end{aligned} \tag{17}$$

$$\begin{aligned}
&(-\alpha_1/s)(\nu_{\theta s}su_{,s}\cos\alpha)-(\alpha_2/s)(\cos\alpha)(u+V_{,\theta}\sin\alpha-W\cos\alpha\csc\alpha)\\
&+(P/2)[(\sin\alpha)(5u+2su_{,s})+4V_{,\theta}+(\sin^2\alpha\sec\alpha)(2W+3W\cos^2\alpha\csc^2\alpha\\
&+4sW_{,s}+2s^2W_{,ss}+3W_{,\theta\theta}\csc^2\alpha)]+(D_1/s^3)(\csc\alpha)[2\nu_{\theta s}W_{,\theta\theta}\\
&+2s^3W_{,sss}\sin^2\alpha-2\nu_{\theta s}sW_{,\theta\theta s}+2\nu_{\theta s}s^2W_{,\theta\theta ss}+s^4W_{,ssss}\sin^2\alpha]\\
&+(D_2/s^3)(\csc\alpha)[sW_{,s}\sin^2\alpha-s^2W_{,ss}\sin^2\alpha+2W_{,\theta\theta}+W_{,\theta\theta\theta\theta}]\\
&+(4D_3/s^3)(\csc\alpha)[W_{,\theta\theta}-sW_{,s\theta\theta}+s^2W_{,ss\theta\theta}]+(Q\sec\alpha)[W_{,ss}]=0
\end{aligned} \tag{18}$$

In addition to the three equations of equilibrium, the following natural boundary conditions are also obtained from Equation (15):

$$\alpha_1 \left( s u_{,s} \sin\alpha + \nu_{\theta s} u \sin\alpha + \nu_{\theta s} v_{,\theta} - \nu_{\theta s} w \cos\alpha \right) - \left(p \sec\alpha/2\right)\left(2 s u \sin^2\alpha + 2 s^2 u_{,s} \sin^2\alpha + s v_{,\theta} \sin\alpha - 2 s w \tan\alpha - s^2 \sin^2\alpha\right) + \left(Q \sec\alpha/2\pi\right)\left(1 - u_{,s}\right)\Big]_{s_1}^{s_0} = 0 \qquad \text{or:} \quad \delta u\Big]_{s_1}^{s_0} = 0 \tag{19 a}$$

$$\left(\alpha_3/s\right)\left(u_{,\theta} \csc\alpha - v + s v_{,s}\right) - \left(P/2\right)\left(3 u_{,\theta} \sec\alpha\right)\Big]_0^{2\pi} = 0 \qquad \text{or:} \quad \delta u\Big]_0^{2\pi} = 0 \tag{19 b}$$

$$\alpha_3\left(u_{,\theta} - v \sin\alpha + s v_{,s} \sin\alpha\right) + \left(p s \sin^2\alpha \sec\alpha\right)\left(v - s v_{,s}\right) + \left(Q \sec\alpha/2\pi s\right)\left(v - s v_{,s}\right)\Big]_{s_1}^{s_0} = 0 \qquad \text{or:} \quad \delta v\Big]_{s_1}^{s_0} = 0 \tag{19 c}$$

$$\alpha_1\left(\nu_{\theta s} u_{,s}\right) + \left(\alpha_2/s\right)\left(u + v_{,\theta} \csc\alpha - w \cot\alpha\right) - \left(P/2\right)\left(4 u \tan\alpha + s u_{,s} \tan\alpha + 3 v_{,\theta} \sec\alpha - 4 w\Big]_0^{2\pi} = 0 \qquad \text{or:} \quad \delta v\Big]_0^{2\pi} = 0 \tag{19 d}$$

$$\left(D_1/s^2\right)\left(-s^2 w_{,ss} \sin\alpha + \nu_{\theta s} w_{,\theta\theta} \cos\alpha - s^3 w_{,sss} \sin\alpha - s \nu_{\theta s} w_{,\theta\theta s} \sec\alpha\right) + \left(D_2/s^2\right)\left(s w_{,s} \sin\alpha + w_{,\theta\theta} \sec\alpha\right) + \left(4 D_3/s^2\right)\left(\csc\alpha\right)\left(w_{,\theta\theta} - s w_{,s\theta\theta}\right) + \left(p s \sin^2\alpha \sec\alpha\right)\left(w - s w_{,s}\right) + \left(Q \sec\alpha/2\pi s\right)\left(w - s w_{,s}\right)\Big]_{s_1}^{s_0} = 0 \qquad \text{or:} \quad \delta w\Big]_{s_1}^{s_0} = 0 \tag{19 e}$$

$$\left(D_1/s\right)\left(-\nu_{\theta s} w_{,ss\theta} \csc\alpha\right) + \left(D_2/s^3\right)\left(s w_{,s\theta} \csc\alpha + w_{,\theta\theta\theta} \csc^3\alpha\right) - \left(4 D_3/s^3\right)\left(\csc\alpha\right)\left(w_{,\theta} - s w_{,s\theta} + s^2 w_{,ss\theta}\right) - \left(P/2\right)\left(3 w_{,\theta} \sec\alpha\right)\Big]_0^{2\pi} = 0 \qquad \text{or:} \quad \delta w\Big]_0^{2\pi} = 0 \tag{19 f}$$

$$\left(D_1/s\right)\left(s^2 w_{,ss} \sin\alpha + \nu_{\theta s} s w_{,s} \sin\alpha + \nu_{\theta s} w_{,\theta\theta} \csc\alpha\right)\Big]_{s_1}^{s_0} = 0 \qquad \text{or:} \quad \delta w_{,s}\Big]_{s_1}^{s_0} = 0 \tag{19 g}$$

$$\left(D_1/s\right)\left(\nu_{\theta s} w_{,ss} \csc\alpha\right) + \left(D_2/s^3\right)\left(s w_{,s} \csc\alpha + w_{,\theta\theta} \csc^3\alpha\right)\Big]_0^{2\pi} = 0 \qquad \text{or:} \quad \delta w_{,\theta}\Big]_0^{2\pi} = 0 \tag{19 h}$$

## A PARTICULAR SOLUTION FOR THE EQUILIBRIUM EQUATIONS AND THE DEVELOPMENT OF A DONNELL TYPE DIFFERENTIAL EQUATION

For convenience the constants in the equilibrium equations are expressed in a non-dimensional form; and, as a result, Equations (16), (17), and (18) are written in the following manner:

$$
\begin{aligned}
&[a_1(S/S_0) + a_2](u/S) + [a_3 + a_4(S/S_0)](u_{,s}) \\
&+[a_5(S_0/S) + a_6 + a_7(S/S_0)](S u_{,ss}) + [a_8(S/S_0) + a_9](u_{,\theta\theta}/S) \\
&+[a_{10}(S/S_0) + a_{11}](v_{,\theta}/S) + [a_{12} + a_{13}(S/S_0)](v_{,s\theta}) \\
&+[a_{14}(S/S_0) + a_{15}](w/S) + [a_{16} + a_{17}(S/S_0)](w_{,s}) = 0
\end{aligned}
\qquad (20)
$$

$$
\begin{aligned}
&[b_1(S/S_0) + b_2](u_{,\theta}/S) + [b_3 + b_4(S/S_0)](u_{,s\theta}) + [b_5(S/S_0) + b_6](v/S) \\
&+[b_7 + b_8(S/S_0)](v_{,s}) + [b_9(S_0/S) + b_{10} + b_{11}(S/S_0)](S v_{,ss}) \\
&+[b_{12}(S/S_0) + b_{13}](v_{,\theta\theta}/S) + [b_{14}(S/S_0) + b_{15}](w_{,\theta}/S) = 0
\end{aligned}
\qquad (21)
$$

$$
\begin{aligned}
&[c_1 + c_2(S/S_0)](u/S) + [c_3 + c_4(S/S_0)](u_{,s}) + [c_5 + c_6(S/S_0)](v_{,\theta}/S) \\
&+c_7(S/S_0) + c_8](w/S) + [c_9(S/S_0) + c_{10}(S_0/S)^2](w_{,s}) \\
&+[c_{11}(S_0/S) + c_{12}(S/S_0) + c_{13}(S_0/S)^2](S w_{,ss}) + [c_{14}(S/S_0) + c_{15}(S_0/S)^2](w_{,\theta\theta}/S) \\
&+[c_{16}(S_0/S)^2](S^2 w_{,sss}) + [c_{17}(S_0/S)^2](w_{,s\theta\theta}) + [c_{18}(S_0/S)^2](S^3 w_{,ssss}) \\
&+[c_{19}(S_0/S)^2](S w_{,ss\theta\theta}) + [c_{20}(S_0/S)^2](w_{,\theta\theta\theta\theta}/S) = 0
\end{aligned}
\qquad (22)
$$

$$
\begin{aligned}
a_1 &= -(3p S_0 \sin^2\alpha)/(2\alpha_1 \cos\alpha) & a_2 &= (\alpha_2/\alpha_1)\sin\alpha \\
a_3 &= -\sin\alpha & a_4 &= (2p S_0 \sin^2\alpha)/(\alpha_1 \cos\alpha) \\
a_5 &= (Q)(2\pi S_0 \alpha_1 \cos\alpha) & a_6 &= -\sin\alpha \\
a_7 &= (p S_0 \sin^2\alpha)/(\alpha_1 \cos\alpha) & a_8 &= (3p\gamma_0)/(2\alpha_1 \cos\alpha) \\
a_9 &= -(\alpha_3)/(\alpha_1 \sin\alpha) & a_{10} &= -(3p S_0 \tan\alpha)/(2\alpha_1) \\
a_{11} &= (\alpha_2 + \alpha_3)/(\alpha_1) & a_{12} &= -(\alpha_1 \gamma_{\theta s} + \alpha_3)/(\alpha_1) \\
a_{13} &= (p S_0 \tan\alpha)(2\alpha_1) & a_{14} &= (3p S_0 \sin\alpha)(2\alpha_1) \\
a_{15} &= -(\alpha_2 \cos\alpha)/(\alpha_1) & a_{16} &= \gamma_{\theta s} \cos\alpha \\
a_{17} &= -(p S_0 \sin\alpha)/(\alpha_1) & &
\end{aligned}
\qquad (23)
$$

$$
\begin{aligned}
b_1 &= (2pS_0 \tan\alpha)/(\alpha_1) & b_2 &= -(\alpha_2+\alpha_3)/(\alpha_1) \\
b_3 &= -(\gamma_{os}\alpha_1+\alpha_3)/(\alpha_1) & b_4 &= (pS_0 \tan\alpha)/(2\alpha_1) \\
b_5 &= -(2pS_0 \sin^2\alpha)/(\alpha_1 \cos\alpha) & b_6 &= (\alpha_3 \sin\alpha)/(\alpha_1) \\
b_7 &= -(\alpha_3 \sin\alpha)/(\alpha_1) & b_8 &= (2pS_0 \sin^2\alpha)/(\alpha_1 \cos\alpha) \\
b_9 &= (Q)(2\pi S_0 \alpha_1 \cos\alpha) & b_{10} &= -(\alpha_3 \sin\alpha)/(\alpha_1) \\
b_{11} &= (pS_0 \sin^2\alpha)/(\alpha_1 \cos\alpha) & b_{12} &= (3pS_0)/(2\alpha_1 \cos\alpha) \\
b_{13} &= -(\alpha_2)/(\alpha_1 \sin\alpha) & b_{14} &= -(2pS_0)/(\alpha_1) \\
b_{15} &= (\alpha_2 \cot\alpha)/(\alpha_1) & &
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
c_1 &= -(\alpha_2 \cos\alpha)/(\alpha_1) & c_2 &= (5pS_0 \sin\alpha)/(2\alpha_1) \\
c_3 &= -\gamma_{os} \cos\alpha & c_4 &= (pS_0 \sin\alpha)/(\alpha_1) \\
c_5 &= -(\alpha_2 \cot\alpha)/(\alpha_1) & c_6 &= (2pS_0)/(\alpha_1) \\
c_7 &= (3pS_0 \cos^2\alpha + 2pS_0 \sin^2\alpha)/(2\alpha_1 \cos\alpha) & c_8 &= (\alpha_2 \cos^2\alpha)/(\alpha_1 \sin\alpha) \\
c_9 &= (2pS_0 \sin^2\alpha)/(\alpha_1 \cos\alpha) & c_{10} &= (h^2 D_2 \sin\alpha)/(12 S_0^2 D_1) \\
c_{11} &= (Q)/(2\pi S_0 \alpha_1 \cos\alpha) & c_{12} &= (pS_0 \sin^2\alpha)(\alpha_1 \cos\alpha) \\
c_{13} &= -(h^2 D_2 \sin\alpha)/(12 S_0^2 D_1) & c_{14} &= (3pS_0)/(2\alpha_1 \cos\alpha) \\
c_{15} &= (2h^2\gamma_{os} D_1 + 2h^2 D_2 + 4h^2 D_3)/(12 S_0^2 D_1 \sin\alpha) & & \\
c_{16} &= (2h^2 \sin\alpha)/(12 S_0^2) & & \\
c_{17} &= -(2\gamma_{os} h^2 D_1 + 4h^2 D_3)/(12 S_0^2 D_1 \sin\alpha) & & \\
c_{18} &= (h^2 \sin\alpha)/(12 S_0^2 & & \\
c_{19} &= (2\gamma_{os} h^2 D_1 + 4h^2 D_3)/(12 S_0^2 D_1 \sin\alpha) & & \\
c_{20} &= (h^2 D_2)/(12 S_0^2 D_1 \sin^3\alpha) & &
\end{aligned}
\tag{23}
$$

Equations (19a) through (19h) are written in the following manner so that the natural boundary conditions are also expressed in a non-dimensional form.

$$
\begin{aligned}
&[a_{22} + a_{21}(S/S_0)^2] + [a_{20}(S/S_0) - 2a_{21}(S/S_0)^2](u/S) \\
&+ [-a_{22} + a_{23}(S/S_0) + a_{24}(S/S_0)^2](u_{,S}) + [a_{25}(S/S_0) + a_{26}(S/S_0)^2](V_{,\theta}/S) \\
&+ [a_{27}(S/S_0) + a_{28}(S/S_0)^2](W/S)\Big\}_{S_1}^{S_0} = 0
\end{aligned}
\tag{19a}
$$

$$[a_{29}(S/S_0)](u_{,\theta}/S)+[a_{30}+a_{31}(S/S_0)+a_{32}(S/S_0)^2](V/S)$$
$$-[a_{30}+a_{31}(S/S_0)+a_{32}(S/S_0)^2](V_{,S})\Big\}_{S_1}^{S_0}=0 \qquad (19c)$$

$$[a_{33}+a_{34}(S/S_0)^2](W/S)+[a_{35}(S_0/S)](W_{,\theta\theta}/S)$$
$$+[a_{36}(S_0/S)+a_{37}+a_{38}(S/S_0)^2](W_{,S})+[a_{39}(S_0/S)](W_{,\theta\theta S}) \qquad (19e)$$
$$+[a_{40}(S_0/S)](SW_{,SS})+[a_{41}(S_0/S)](S^2W_{,SSS})\Big\}_{S}^{S_0}=0$$

$$a_{42}(W_{,\theta\theta}/S)+a_{43}(W_{,S})+a_{44}(SW_{,SS})\Big\}_{S}^{S_0}=0 \qquad (19g)$$

$$[b_{20}+b_{21}(S/S_0)](u_{,\theta}/S)+b_{22}(V/S)+b_{23}(V_{,S})\Big\}_{0}^{2\pi}=0 \qquad (19b)$$

$$[b_{24}+b_{25}(S/S_0)](u/S)+[b_{26}+b_{27}(S/S_0)](V_{,\theta}/S)$$
$$+[b_{28}+b_{29}(S/S_0)](W/S)+[b_{30}+b_{31}(S/S_0)](u_{,S})\Big\}_{0}^{2\pi}=0 \qquad (19d)$$

$$[b_{32}(S_0/S)^2+b_{33}(S/S_0)](W_{,\theta}/S)+[b_{34}(S_0/S)^2](W_{,\theta\theta\theta}/S)$$
$$+[b_{35}(S_0/S)^2](W_{,S\theta})+[b_{36}(S_0/S)^2](SW_{,SS\theta})\Big\}_{0}^{2\pi}=0 \qquad (19f)$$

$$[b_{37}(S_0/S)^2](W_{,\theta\theta}/S)+[b_{38}(S_0/S)^2](W_{,S})+[b_{39}(S_0/S)^2](S_0/S)^2](SW_{,SS})\Big\}_{0}^{2\pi}=0 \qquad (19h)$$

where

$a_{20}=\gamma_{\theta S}$ $\qquad a_{21}=(pS_0\tan\alpha)/(2\alpha_1)$

$a_{22}=(Q\sec\alpha\csc\alpha)/(2\pi\alpha_1S_0)$ $\qquad a_{23}=1$

$a_{24}=-(pS_0\tan\alpha)/(\alpha_1)$ $\qquad a_{25}=\gamma_{\theta S}\csc\alpha$

$a_{26}=-(pS_0\sec\alpha)/(2\alpha_1)$ $\qquad a_{27}=-\gamma_{\theta S}\cot\alpha$

$a_{28}=(pS_0)/(\alpha_1)$ $\qquad a_{29}=(\alpha_3\csc\alpha)/(\alpha_1)$ $\qquad (24)$

$a_{30}=(Q\sec\alpha\csc\alpha)/(2\pi S_0\alpha_1)$ $\qquad a_{31}=-(\alpha_3)/(\alpha_1)$

$a_{32}=(pS_0\tan\alpha)/(\alpha_1)$ $\qquad a_{33}=(Q\sec\alpha)/(2\pi S_0\alpha_1)$

$a_{34}=(pS_0\sin^2\alpha\sec\alpha)/(\alpha_1)$

$a_{35}=(h^2\gamma_{\theta S}D_1\cos^2\alpha+h^2D_2+4h^2D_3\csc\alpha\cos\alpha)/(12S_0^2D_1\cos\alpha)$

$a_{36}=(h^2D_2\sin\alpha)/(12D_1S_0^2)$ $\qquad a_{37}=(-Q\sec\alpha)/(2\pi S_0\alpha_1)$

$a_{38}=-(pS_0\sin^2\alpha\sec\alpha)/(\alpha_1)$

$$a_{39} = -(h^2 D_1 \gamma_{os} \sec\alpha + 4h^2 D_3 \cos\alpha)/(12 S_0^2 D_1)$$

$$a_{40} = -(h^2 \sin\alpha)/(12 S_0^2) \qquad a_{41} = -(h^2 \sin\alpha)/(12 S_0^2) \tag{24}$$

$$a_{42} = \gamma_{os} \cos\alpha \qquad a_{43} = \gamma_{os} \sin\alpha \qquad a_{44} = \sin\alpha$$

$$b_{20} = (\alpha_3 \cos\alpha)/(\alpha_1) \qquad b_{21} = -(3p S_0 \sec\alpha)/(2\alpha_1)$$

$$b_{22} = -(\alpha_3/\alpha_1) \qquad b_{23} = (\alpha_3/\alpha_1)$$

$$b_{24} = \alpha_2/\alpha_1 \qquad b_{25} = -(2p S_0 \tan\alpha)/(\alpha_1)$$

$$b_{26} = (\alpha_2 \cos\alpha)/(\alpha_1) \qquad b_{27} = -(3p S_0 \sec\alpha)/(2\alpha_1)$$

$$b_{28} = -(\alpha_2 \cot\alpha)/(\alpha_1) \qquad b_{29} = (2p S_0)/(\alpha_1) \tag{24}$$

$$b_{30} = \gamma_{os} \qquad b_{31} = -(p S_0 \tan\alpha)/(2\alpha_1)$$

$$b_{32} = -(D_3 h^2 \cos\alpha)/(3 S_0^2 D_1) \qquad b_{33} = -(3p S_0)/(2\alpha_1)$$

$$b_{34} = (D_2 h^2 \cos^3\alpha)/(12 S_0^2 D_1) \qquad b_{35} = [(D_2 h^2 + 4 D_3 h^2)\cos\alpha]/(12 S_0^2 D_1)$$

$$b_{36} = -(\gamma_{os} h^2 D_1 \cos\alpha + 4h^2 D_3 \cos\alpha)/(12 S_0^2 D_1)$$

$$b_{37} = (D_2 \cos^3\alpha)/(D_1) \qquad b_{38} = (D_2 \cos\alpha)/(D_1)$$

$$b_{39} = \gamma_{os} \cos\alpha$$

Let $S_0$ be designated as the value of $S$ at the base of the cone frustum. Then a non-dimensional coordinate in the generatrix direction is defined in the following manner:

$$\bar{S} = \frac{S}{S_0} \tag{25}$$

Equations (16), (17), and (18) can be reduced to ordinary differential equations by making the following substitutions:

$$u = S_0 \bar{F} \sin n\theta \tag{26}$$

$$v = S_0 \bar{G} \cos n\theta \tag{27}$$

$$w = S_0 \bar{H} \sin n\theta \tag{28}$$

In the preceding expressions $n$ is an integer and represents the number of buckling waves in the circumferential direction; and $\bar{F}$ , $\bar{G}$ , and $\bar{H}$ are non-dimensional functions of $\bar{S}$ .

After the substitution of Equations (25) to (28) into Equations (20), (21), and (22) and after all the indicated operations have been performed, the following differential equations in $\bar{F}$ , $\bar{G}$ , and $\bar{H}$ result:

$$
\begin{aligned}
&[(a_2 - a_9 n^2)/(\bar{s}) + (a_1 - a_8 n^2)]\bar{F} + [a_3 + a_4\bar{s}]\frac{d\bar{F}}{d\bar{s}} \\
&+ [a_5 + a_6\bar{s} + a_7\bar{s}^2]\frac{d^2\bar{F}}{d\bar{s}^2} - [a_{11}n/(\bar{s}) + a_{10}n]\bar{G} \\
&- [a_{12}n + a_{13}n\bar{s}]\frac{d\bar{G}}{d\bar{s}} + [(a_{15})/(\bar{s}) + a_{14}]\bar{H} \\
&+ [a_{16} + a_{17}\bar{s}]\frac{d\bar{H}}{d\bar{s}} = 0
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&[(b_2 n)/(\bar{s}) + b_1 n]\bar{F} + [b_3 n + b_4 n\bar{s}]\frac{d\bar{F}}{d\bar{s}} \\
&+ [b_6 - b_{13}n^2)/(\bar{s}) + (b_5 - b_{12}n^2)]\bar{G} + [b_7 + b_8\bar{s}]\frac{d\bar{G}}{d\bar{s}} \\
&+ [b_9 + b_{10}\bar{s} + b_{11}\bar{s}^2]\frac{d^2\bar{G}}{d\bar{s}^2} + [(b_{15}n)/(\bar{s}) + b_{14}n]\bar{H} = 0
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&[(c_1)/(\bar{s}) + c_2]\bar{F} + [c_3 + c_4\bar{s}]\frac{d\bar{F}}{d\bar{s}} - [(c_5 n)/(\bar{s}) + nc_6]\bar{G} \\
&+ [(c_{20}n^4 - c_{15}n^2)/(\bar{s}^3) + (c_8)/(\bar{s}) + c_7 - c_{14}n^2]\bar{H} \\
&+ [(c_{10} - c_{17}n^2)/(\bar{s}^2) + c_9\bar{s}]\frac{d\bar{H}}{d\bar{s}} + [(c_{13} - c_{19}n^2)/(\bar{s}) + c_{11} + c_{12}\bar{s}^2]\frac{d^2\bar{H}}{d\bar{s}^2} \\
&+ [c_{16}]\frac{d^3\bar{H}}{d\bar{s}^3} + [c_{18}\bar{s}]\frac{d^4 H}{d\bar{s}^4} = 0
\end{aligned}
\tag{31}
$$

It is easily seen that the functions assumed in Equations (26), (27), and (28) satisfy the boundary conditions given by Equations (19b), (19d), (19f), and (19h). The substitution of Equations (26), (27), and (28) into Equations (19a), (19c), (19e), and (19g) results in the following expressions for the natural boundary conditions that contain only the dimensionless coordinate .

$$
\begin{aligned}
&(a_{22} + a_{21}\bar{s}^2) + (a_{20} - 2a_{21}\bar{s})\bar{F} + (-a_{22} + a_{23}\bar{s} + a_{24}\bar{s}^2)\frac{d\bar{F}}{d\bar{s}} \\
&- (na_{25} + na_{26}\bar{s})\bar{G} + (a_{27} + a_{28}\bar{s})\bar{H}\Big]_{s_1}^{s_0} = 0
\end{aligned}
\tag{32}
$$

$$
(na_{29})\bar{F} - (a_{30}/\bar{s} + a_{31} + a_{32}\bar{s})\bar{G} + (a_{30} + a_{31}\bar{s} + a_{32}\bar{s}^2)\frac{d\bar{G}}{d\bar{s}}\Big\}_{s_1}^{s_0} = 0 \tag{33}
$$

$$
\begin{aligned}
&[(a_{33}/\bar{s}) - n^2 a_{35} + a_{34}\bar{s}]\bar{H} + [(a_{36} - n^2 a_{39})/\bar{s} + a_{37} + a_{38}\bar{s}^2]\frac{d\bar{H}}{d\bar{s}} \\
&+ [a_{40}\bar{s}^2]\frac{d^2\bar{H}}{d\bar{s}^2} + [a_{41}\bar{s}]\frac{d^3\bar{H}}{d\bar{s}^3}\Big\}_{s_1}^{s_0} = 0
\end{aligned}
\tag{34}
$$

$$\left.\left[(n^2 a_{42})/(\bar{s})\right]\bar{H} - [a_{43}]\frac{d\bar{H}}{d\bar{s}} - [a_{44}\bar{s}]\frac{d^2\bar{H}}{d\bar{s}^2}\right\}_{s_1}^{s_0} = 0 \tag{35}$$

After very lengthy mathematical manipulations, Equations (29), (30), and (31) can be reduced to the following Donnell-type eithth order differential equation in terms of the function :

$$\sum_{i=0}^{i=8}\sum_{j=i-39}^{j=i+23} T_{ij}\, s^j \frac{d^i\bar{H}}{ds^i} - \sum_{i=7}^{i=8}\sum_{j=i+21}^{j=i+23} T_{ij}\, s^j \frac{d^i\bar{H}}{ds^i} = 0 \tag{36}$$

where the are constants that are functions of the stiffness coefficients $\alpha_1$, $\alpha_2$, and $\alpha_3$, the rigidity coefficients $D_1$, $D_2$, and $D_3$, the apex angle $\alpha$, the applied pressure $p$, and the axial load $Q$.

Because the constant coefficients, $T_{ij}$, are extremely long, they are included in a separate appendix. This appendix can be obtained by writing to the Bureau of Engineering Research, P. O. Box 6162, University, Alabama.

## CONCLUSION

At first glance it would seem to require a prohibitive amount of time to obtain practical results from Equation (36). The use of a high speed digital computer and the application of an approximate method, such as the Galerkin method employed in Reference (5), could possibly yield meaningful practical results in a reasonable amount of time when such results are highly desirable. A solution of the type suggested in Reference (6) might also prove feasible.

The effort of this paper clearly shows the great mathematical difficulty posed by the problem of the determination of the instability criteria for a stiffened cone frustum subjected to a combination of loads. In connection with this problem, and the problem posed in Reference (2), a very serious need exists for the mathematical investigation of the solution of an eighth order linear homogenous differential equation with variable coefficients and the determination of the physical instability criteria represented by such an equation.

NOTATION

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|---|---|
| $a_1, b_1, c_1$, etc. | Constants that are functions of the extensional and shear stiffnesses, the bending and twist rigidities, the applied pressure, and the axial load. |
| $A_s$ | Area of the middle surface of the shell |
| $D_1, D_2, D_3$ | Bending and twist rigidities of an elemental area of an orthotropic circular conical shell |
| $e_{ss}, e_{\theta\theta}, e_{\theta s}$ | Generatrix, circumferential and shearing strain |
| $E_s, E_\theta$ | Moduli of elasticity for orthotropic circular conical shell |
| $\bar{F}, \bar{G}, \bar{H}$ | Functions of the non-dimensional coordinate in the generatrix direction |
| $G$ | Shear modulus for orthotropic circular conical shell |
| $h$ | Wall thickness of circular conical shell |
| $n$ | Integer that indicates buckled mode in the circumferential direction |
| $\bar{N}_{ss}, \bar{N}_{\theta\theta}, \bar{N}_{\theta s}$ | Generatrix, circumferential and shear stress resultants per unit length |
| $p$ | Radial or hydrostatic pressure |
| $Q$ | Axial compressive load |
| $s, \theta, z$ | Generatrix, circumferential and radial coordinates of cone middle surface |
| $s_0, s_1$ | Generatrix coordinates of the base and the top of circular cone frustum |
| $\bar{s}$ | Non-dimensional coordinate in the generatrix direction |
| $u, v, w$ | Generatrix, circumferential and radial displacements of cone middle surface |
| $U$ | Change in the strain energy of the shell during the buckling process |
| $V$ | Change in the potential energy of the external forces during the buckling process |

| | |
|---|---|
| $V_c$ | Volume enclosed by the shell |
| $V_s$ | Volume of the shell wall |
| $2\alpha$ | Apex angle of cone or cone frustum extended |
| $\alpha_1, \alpha_2, \alpha_3$ | Extensional and shearing stiffnesses of orthotropic conical shell |
| $\nu_{s\theta}, \nu_{\theta s}$ | Poisson's ratios for orthotropic shell |
| $\bar{\sigma}_{ss}, \sigma_{ss}, \bar{\sigma}_{\theta\theta}, \sigma_{\theta\theta}, \bar{\sigma}_{\theta s}, \sigma_{\theta s}$ | Generatrix, circumferential and shearing stresses |

## REFERENCES


(1) S. R. Bodner, "General Instability of a Ring-Stiffened Circular Cylindrical Shell Under Hydrostatic Pressure," J. Appl. Mech., pp. 269-277 (June, 1957)

(2) C.C. Steyer and S. Zien, "General Instability of an Orthotropic Circular Conical Shell Under Hydrostatic Pressure," Proceedings of the Seventh Midwestern Mechanics Conference, Developments in Mechanics, Vol. 1, edited by J.E. Lay and L. E. Malvern, pp. 153-176 (Sept., 1961)

(3) F.I.N. Noirdson, "Buckling of Conical Shells Subjected to Uniform External Pressure," Trans. Roy. Inst. Tech., Stockholm, Sweden, No. 10. (1947).

(4) P. Seide, "Axisymmetrical Buckling of Circular Cones Under Axial Compression," J. Appl. Mech., Trans. ASME, Vol. 78, pp. 625-628. (1956)

(5) S. Y. Lu and W. A. Nash, "Elastic Instability of Pressurized Cylindrical Shells Under Compression or Bending," Tech. Rep. No. 1 for Res. Grant NsG-16-59, NASA; Engr. and Ind. Exp. Stat., University of Florida, (January, 1962).

(6) N.J. Hoff, "Thin Circular Conical Shells Under Arbitrary Loads," J. Appl. Mech., pp. 557-562 (December, 1955).